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PROBABILITY, STATISTICS AND RANDOM PROCESSES

This course is an introduction to probability, statistics and random processes

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I. POBABILITY

Basic Ideas of Probability

1. Probability Spaces

There are two definitions of probabilities for random events: classical and modern. The modern definition of probability is based on the measure theory in which a random event is nothing but a set and its probability is the measure of the set.

Definition (Sigma-Algebra) Let Ω be a set and Π a class Π of subsets of Ω, i.e., a subset of 2^{Ω}, Π is said to be a $\sigma-$*algebra* of Ω if

(1) $\Omega \in \Pi$

(2) if $A \in \Pi$, then $\overline{A} = \Omega - A \in \Pi$ (which implies that $\phi \in \Pi$)

(3) if $A_i \in \Pi$, where $i \in I$ and I is at most a countable index set, then $\bigcup_{i \in I} A_i \in \Pi$ (which means that the class Π is closed with respect to union)

Remark 1: 2^{Ω} is the power set of Ω, i.e., the set of all subsets of Ω.

Remark 2: In measure theory, (Ω, Π) is called a *measurable space*.

Remark 3: Since $\bigcap_{i \in I} A_i = \overline{\overline{\bigcap_{i \in I} A_i}} = \overline{\bigcup_{i \in I} \overline{A_i}} \in \Pi$, Π is also closed with respect to intersection.

Example Let $\Omega = \{\omega_1, \omega_2\}$, $\Pi = \{\phi, \{\omega_1\}, \{\omega_2\}, \{\omega_1, \omega_2\}\}$, where ϕ stands for empty set, Π is then a $\sigma - a\lg ebra$.

Definition (Probability Space) Let Ω be a set, Π a σ-algebra of Ω and P a real-valued function defined on Π, the triplet (Ω, Π, P) is called a *probability space* if P satisfies the following conditions

(1) $P(A) \geq 0$ for all $A \in \Pi$

(2) $P\left(\bigcup_{i=1}^{+\infty} A_i\right) = \sum_{i=1}^{+\infty} P(A_i)$ for all $A_1, A_2, \cdots, A_n, \cdots \in \Pi$ such that $A_i \cap A_j = \phi$ when $i \neq j$

(3) $P(\Omega) = 1$ (which implies that $P(\phi) = 0$)

Remark 1: Usually, Ω is often called *sample space*, Π the field of *random events* and for all $A \in \Pi$, $P(A)$ the *probability* of occurrence of A.

Remark 2: In measure theory, the probability space (Ω, Π, P) is also called *measured space*.

Remark 3: Two random events A and B are said to be *incompatible* if $AB = \phi$. In this case, $P(AB) = 0$.

1.1. Discrete Probability Spaces

The number of all possible occurrences in a random experiment is countable.

Definition A probability space (Ω, Π, P) is called a *discrete probability space* if the sample space Ω is a countable (finite or denumerable infinite) set and $\Pi = 2^{\Omega}$.

Remark 1: To specify a discrete probability P, it suffices to specify a mapping $p : \Omega \to [0,1]$ such that $p(\omega) \geq 0$ for all $\omega \in \Omega$ and $\sum_{\omega \in \Omega} p(\omega) = 1$. Then, for all $A \in \Pi$, $P(A) = \sum_{\omega \in A} p(\omega)$.

Remark 2: If $\Omega = \{\omega_1, \omega_2, \cdots, \omega_N\}$ and $p(\omega_i) = \frac{1}{N}$, where $i = 1,2,\cdots,N$, then the resulting triple (Ω, Π, P) is called *classical* probability space.

Example Let $\Omega = \{\omega_1, \omega_2\}$, $\Pi = \{\phi, \{\omega_1\}, \{\omega_2\}, \{\omega_1, \omega_2\}\}$, and

(1) $p(\omega_1) = \frac{1}{3}$, $p(\omega_2) = \frac{2}{3}$, then (Ω, Π, P) is a discrete probability space

(2) $p(\omega_1) = p(\omega_2) = \frac{1}{2}$, then (Ω, Π, P) is a classical probability space

Example Let $\Omega=\{\omega_1,\omega_2,\cdots,\omega_n,\cdots\}$, $\Pi=2^{\Omega}$ and $p(\omega_n)=\dfrac{\frac{1}{n^2}}{\sum_{k=1}^{+\infty}\frac{1}{k^2}}=\dfrac{6}{(n\pi)^2}$, $n=1,2,\cdots$, then (Ω,Π,P) is a discrete probability space.

1.2. Continuous Probability Spaces

The number of all possible occurrences in a random experiment is uncountable.

Definition A probability space (Ω,Π,P) is called a *continuous probability space* if the sample space Ω is a continuum.

Example (Geometric Probability) Assume that the sample Ω is an interval, an area or a volume, then the probability of a point falling into a part of Ω is given by

$$P=\frac{\text{Measure of the part of }\Omega}{\text{Measure of }\Omega}$$

1.3. Properties of Probability

Theorem (Finite Measure) Let (Ω,Π,P) be a probability space, then for all $A\in\Pi$,

$$P(A)+P(\overline{A})=P(\Omega)=1\ \Rightarrow\ P(A)\le 1$$

Theorem (Monotonicity) Let (Ω,Π,P) be a probability space, then for all $A,B\in\Pi$,

$$A\subseteq B\ \Rightarrow\ P(A)\le P(A)+P(B-A)=P(B)$$

Theorem (Union) Let (Ω,Π,P) be a probability space, then for all $A,B\in\Pi$,

$$P(A \cup B) = P(A \cup (B - A)) = P(A) + P(B - A) = P(A) + P(B) - P(A \cap B)$$

Theorem (Union) Let (Ω, Π, P) be a probability space, then for all $A_1, A_2, \cdots, A_n \in \Pi$,

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le n} P\left(A_{i_1} \cdots A_{i_k}\right)$$

Hint:

$$P\left(\bigcup_{i=1}^{n+1} A_i\right) = P\left(\bigcup_{i=1}^{n} A_i\right) + P(A_{n+1}) - P\left(\bigcup_{i=1}^{n} (A_i A_{n+1})\right)$$

$$= \sum_{k=1}^{n} \left\{(-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le n} P\left(A_{i_1} \cdots A_{i_k}\right)\right\} + P(A_{n+1}) - \sum_{k=1}^{n} \left\{(-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le n} P\left(A_{i_1} \cdots A_{i_k} A_{n+1}\right)\right\}$$

$$= \sum_{1 \le i_1 \le n} P\left(A_{i_1}\right) + P(A_{n+1})$$

$$+ \sum_{k=2}^{n} \left\{(-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le n} P\left(A_{i_1} \cdots A_{i_k}\right)\right\} + \sum_{k=1}^{n-1} \left\{(-1)^{k} \sum_{1 \le i_1 < \cdots < i_k \le n} P\left(A_{i_1} \cdots A_{i_k} A_{n+1}\right)\right\}$$

$$+ (-1)^n \sum_{1 \le i_1 < \cdots < i_n \le n} P\left(A_{i_1} \cdots A_{i_n} A_{n+1}\right)$$

$$= \sum_{1 \le i \le n+1} P(A_i)$$

$$+ \sum_{k=2}^{n} (-1)^{k-1} \left\{ \sum_{1 \le i_1 < \cdots < i_k \le n} P\left(A_{i_1} \cdots A_{i_k}\right) + \sum_{1 \le i_1 < \cdots < i_{k-1} \le n} P\left(A_{i_1} \cdots A_{i_{k-1}} A_{n+1}\right)\right\}$$

$$+ (-1)^n P(A_1 \cdots A_n A_{n+1})$$

$$= \sum_{1 \le i \le n+1} P(A_i) + \sum_{k=2}^{n} \left\{(-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le n+1} P\left(A_{i_1} \cdots A_{i_k}\right)\right\} + (-1)^n P(A_1 \cdots A_n A_{n+1})$$

$$= \sum_{k=1}^{n+1} \left\{(-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le n+1} P\left(A_{i_1} \cdots A_{i_k}\right)\right\}$$

2. Conditional Probability and Statistical Independence

2.1. Conditional Probability

Definition Let (Ω, Π, P) be a probability space and $A, B \in \Pi$, the *conditional probability* of B, given that A has occurred, is defined as $P(B/A) = \frac{P(AB)}{P(A)}$, where $P(A) > 0$.

Theorem Let (Ω, Π, P) be a probability space and $A \in \Pi$ with $P(A) > 0$, the triplet (Ω_A, Π_A, P_A) is also a probability space, where $\Omega_A = \Omega \cap A$, $\Pi_A = \{AB | B \in \Pi\}$ and $P_A(AB) = P(B/A)$.

2.2. Composite Probability Formulae

Theorem (Composite Probability Formula) Let (Ω, Π, P) be a probability space, and $A \in \Pi$, if $A \subseteq \bigcup_k E_k$, where $E_k \in \Pi$ with $P(E_k) > 0$ and $E_i \cap E_j = \phi$ for all $i \neq j$, then $P(A) = \sum_k P(A/E_k) P(E_k)$.

Proof:

$$P(A) = P\left(A\left(\bigcup_k E_k\right)\right) = P\left(\bigcup_k (AE_k)\right) = \sum_k P(AE_k) = \sum_k P(A/E_k)P(E_k) \ \#$$

Remark:

$$A \subseteq B \Rightarrow A \cap B = A, \ A \cap \left(\bigcup_k E_k\right) = \bigcup_k (A \cap E_k)$$

2.3. Bayes Formulae

Theorem (Bayes Formula) Let (Ω, Π, P) be a probability space and $A \in \Pi$ with $P(A) > 0$, if $A \subseteq \bigcup_k E_k$, where $E_k \in \Pi$ with $P(E_k) > 0$ and $E_i \cap E_j = \phi$ for all $i \neq j$, then

$$P(E_i/A) = \frac{P(E_i)P(A/E_i)}{\sum_k P(E_k)P(A/E_k)}.$$

Proof:

$$P(E_i/A) = \frac{P(AE_i)}{P(A)} = \frac{P(E_i)P(A/E_i)}{\sum_k P(E_k)P(A/E_k)} \quad \#$$

2.4. Statistical Independence

Definition Let (Ω, Π, P) be a probability space and $A, B \in \Pi$, A and B are said to be *statistically independent* if $P(AB) = P(A)P(B)$.

Remark 1: If A and B are independent, then $P(A/B) = \frac{P(AB)}{P(B)} = P(A)$.

Remark 2: Recall that two events A and B are said to be incompatible if $AB = \phi$. In this case, $P(AB) = 0$.

Definition Let (Ω, Π, P) be a probability space and Π' a subset of Π, Π' is said to be *statistically independent* if for all finite subsets Π'' of Π', $P\left(\bigcap_{A \in \Pi''} A\right) = \prod_{A \in \Pi''} P(A)$.

Remark: The statistical independence of any two events of Π' can not guarantee the statistical independence of Π'. For example, $\Pi' = \{A, B, C\}$, Π' is statistically independent if

$$P(AB) = P(A)P(B),\ P(AC) = P(A)P(C),\ P(BC) = P(B)P(C),\ P(ABC) = P(A)P(B)P(C)$$

are established at the same time.

Appendix Combinatorics

Sample Selection Suppose there are m distinguishable elements, how many ways there are in which one can select r elements from these m distinguishable elements?

Order counts?	Repetitions are allowed? (With/Without replacement)	The number of ways to choose the samples	Remarks
Yes	Yes	m^r	Permutation
Yes	No	$\frac{m!}{(m-r)!}$	Permutation
No	Yes	$\frac{(m+r-1)!}{r!(m-1)!}$	Combination
No	No	$\frac{m!}{r!(m-r)!}$	Combination

Balls into Cells There are eight different ways in which n balls can be placed into k cells:

Distinguish the balls?	Distinguish the cells?	Can cells be empty?	The number of ways to place n balls into k cells
Yes	Yes	Yes	k^n
Yes	Yes	No	$k!\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$
No	Yes	Yes	$\frac{(k+n-1)!}{n!(k-1)!}$
No	Yes	No	$\frac{(n-1)!}{(k-1)!(n-k)!}$

Yes	No	Yes	$\sum_{r=1}^{k}\left\{\begin{matrix} n \\ r \end{matrix}\right\}$
Yes	No	No	$\left\{\begin{matrix} n \\ k \end{matrix}\right\}$
No	No	Yes	$\sum_{r=1}^{k} p_r(n)$
No	No	No	$p_k(n)$

where $\left\{\begin{matrix} n \\ k \end{matrix}\right\} = \frac{1}{k!}\sum_{r=1}^{k}(-1)^{k-r}\binom{k}{r}r^n$ is the Stirling cycle number and $p_k(n)$ the number of partition of the number n into exactly k integer pieces.